Motivic degree zero DT-invariants

(Behrend, Bryan, Szendrői)

<u>Introduction</u>: Recall from Bochao's talk that if X is a scheme (finite type/$\mathbb{C}$), the virtual Euler characteristic of X is

$$\chi_{vir}(X) := \sum_{k \in \mathbb{Z}} k \cdot \underbrace{\chi}_{\text{regular Euler characteristic}}(\underbrace{\nu_X^{-1}}_{\text{Behrend function of } X}(k))$$

This is sensitive to singularities,

scheme structure (important for enumerative geometry in dim ≥ 3).

For X a Calabi-Yau threefold, we saw that $\chi_{vir}(\mathrm{Hilb}^n X)$ is the degree zero DT invariant of X.

Can this come from something "greater" (i.e. "motivic")?

If X is a $\mathbb{C}$-variety, a "virtual motive" of X is a class $[X]_{vir}$ in "ring of motivic weights" such that

$$\chi([X]_{vir}) = \chi_{vir}(X).$$

(need to define for motivic weight)

We will construct a virtual motive for $Hilb^n X$ for X a threefold over $\mathbb{C}$ (not necessarily Calabi-Yau), and because of motivation above call $[Hilb^n X]_{vir}$ a motivic degree zero Donaldson-Thomas invariant.

This will use description of

$\mathrm{Hilb}^n \mathbb{C}^3$ as a degeneracy locus, and will give motivic Göttsche-like formulas for partition functions of threefolds.

What is the ring of motivic weights?

Let us first define the Grothendieck ring of varieties:

$K_0(\mathrm{Var}_{\mathbb{C}})$ is the free abelian group on isomorphism classes of varieties over $\mathbb{C}$ with product given by taking Cartesian products modulo scissor relations:

- if $Y \subset X$ is a closed subvariety,

$$[X] = [Y] + [X \setminus Y]$$

in $K_0(\mathrm{Var}_{\mathbb{C}})$.

Some standard facts:

- if $F \to E \to B$ is a Zariski-trivial fibration then

$$[E] = [F] \cdot [B]$$

(since B is a variety, can use quasicompactness to construct finite cover by trivializing open sets, then induct on # of trivializing opens and use scissor relations cleverly)

- if X is stratified by disjoint locally closed subsets as $X = \bigsqcup_{i=1}^{n} X_i$ then

$$[X] = \sum_{i=1}^{n} [X_i]$$

(... requires

(follows from scissor relation but requires an argument)

- if $f: X \to Y$ is a bijective morphism on $\mathbb{C}$-points then $[X]=[Y]$ (proof requires stratification property).

This last point can be used to show that $K_0(Var_{\mathbb{C}}) \cong K_0(Sch_{\mathbb{C}}) = K_0(Sp_{\mathbb{C}})$ where $Sch_{\mathbb{C}}$, $Sp_{\mathbb{C}}$ are the categories of schemes and algebraic spaces respectively.

To get ring of motivic weights, just

define

$$\mathcal{M}_{\mathbb{C}} = K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-\frac{1}{2}}]$$

formally, where $\mathbb{L} = [\mathbb{A}^1]$

("Lefschetz motive").

Computations in $\mathcal{M}_{\mathbb{C}}$ are very concrete using fibration property.

Examples:

For notation, set $[n]_{\mathbb{L}}! = (\mathbb{L}^n - 1)\cdot(\mathbb{L}^{n-1} - 1) \cdots (\mathbb{L} - 1)$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{L}} = \frac{[n]_{\mathbb{L}}!}{}.$$

$\mathbb{L}$

$$\frac{\qquad\qquad\qquad}{[k]_{\mathbb{L}}!\,[n-k]_{\mathbb{L}}!}.$$

1) $[GL_n(\mathbb{C})] = \mathbb{L}^{\binom{n}{2}}[n]_{\mathbb{L}}!$

$$= (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L})\cdots(\mathbb{L}^n - \mathbb{L}^{n-1}),$$

since can build an element of GL_n by first choosing a nonzero column $(\mathbb{L}^n - 1)$, then choosing an independent second column $(\mathbb{L}^n - \mathbb{L})$, etc.

2) $[Gr(k,n)] = \begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{L}}.$

Using identity $\binom{n}{2} - \binom{k}{2} - \binom{n-k}{2}$

$= k(n-k)$, equivalent to showing

$$[Gr(k,n)] = \frac{[GL_n]}{\mathbb{L}^{k(n-k)}[GL_k][GL_{n-k}]}.$$

$Gr(k,n)$ has GL_n-action, and

if $\Lambda \in Gr(k,n)$ is $\{x_{k+1} = \dots = x_n = 0\}$,

then stabilizer of Λ explicitly is

matrices of form

$$\begin{pmatrix} GL_k & \mathbb{A}_{k,n-k} \\ 0_{n-k,k} & GL_{n-k} \end{pmatrix},$$

so follows from fibration property.

The following is known:

(Morrison-Bryan, Feit-Fine) Let C_n denote the reduced variety of pairs of commuting matrices in $\mathrm{End}(\mathbb{C}^n)^{\times 2}$. Then

$$\sum_{n \geq 0} \frac{[C_n]}{[GL_n]} t^n =$$

$$\prod_{m=1}^{\infty} \prod_{j=0}^{\infty} \left(1 - \mathbb{L}^{1-j} t^m\right)^{-1} \quad \text{in}$$

$$\mathcal{M}_{\mathbb{C}}\left[(1-\mathbb{L}^n)^{-1} : n \geq 1\right].$$

Specializations of motivic classes:

Can recover lots of invariants of varieties via homomorphisms from $\mathcal{M}_{\mathbb{C}}$ to other rings.

Deligne's mixed Hodge structure gives
E-polynomial homomorphism

$$E : \mathcal{M}_{\mathbb{C}} \longrightarrow \mathbb{Z}\left[x, y, (xy)^{-\frac{1}{2}}\right]$$

[illegible]

X variety $\longmapsto \sum_{p,q} x^p y^q \sum_i (-1)^i \dim H^{p,q}(H^i(X,\mathbb{Q}))$

$\mathbb{L}^{-\frac{1}{2}} \longmapsto (xy)^{-\frac{1}{2}}$

Specialize $x = y = -q^{1/2}$, $(xy)^{1/2} = q^{1/2}$,

get weight polynomial

$W : \mathcal{M}_{\mathbb{C}} \longrightarrow \mathbb{Z}[q^{\pm\frac{1}{2}}]$,

which just gives Poincaré polynomial in variable $q^{\frac{1}{2}}$ for X smooth projective.

Finally, setting $q^{\frac{1}{2}} = -1$ gives

Euler characteristic

$$\chi : \mathcal{M}_{\mathbb{C}} \to \mathbb{Z}.$$

Power structure for motivic weights:

A power structure on a ring R is a map

$$R \times (1 + tR[[t]]) \to 1 + tR[[t]]$$

$$(m, A(t)) \mapsto A(t)^m$$

satisfying some usual exponential properties

$(A(t)^0 = 1,\ A(t)^1 = A(t),\ A(t)^{m+n} = A(t)^m \cdot A(t)^n,$ etc...)

For $R = K_0(\mathrm{Var}_{\mathbb{C}})$, Gusein-Zade et al constructed a power structure uniquely characterized by

$$(1-t)^{-[X]} = \sum_{n=0}^{\infty} [\mathrm{Sym}^n X]\, t^n$$

for X a variety. Briefly state construction:

if $A(t) = 1 + \sum_{i \geq 1} A_i t^i$, X variety,

$$A(t)^{[X]} = 1 + \sum_{\alpha \text{ partition}} \pi_{G_\alpha} \left[\left(\prod_i X^{\alpha_i} \setminus \Delta \right) \cdot \left(\prod_i A_i^{\alpha_i} \right) \right] t^{|\alpha|}$$

where α runs over all partitions of all lengths,

$\alpha = 1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} \ldots$, Δ is the "big diagonal" in X^{α_i} (all pts in product where some pair of factors is equal, so $X^{\alpha_i} \setminus \Delta$ is tuples of distinct points), and π_{G_α} maps the input to the orbit space under the permutation action by $G_\alpha := \prod_i S_{\alpha_i}$.

Can extend in straightforward way to power structure on $\mathcal{M}_{\mathbb{C}}$.

Also have Exp map

$$\mathrm{Exp}: t\mathcal{M}_{\mathbb{C}}[[t]] \longrightarrow 1 + t\mathcal{M}_{\mathbb{C}}[[t]]$$

$$\sum_{n=1}^{\infty} [A_n] t^n \mapsto \prod_{n=1}^{\infty} (1-t^n)^{-[A_n]}.$$

Definitions seem weird, but make sense for partition functions:

For X a variety, define

$$H_X(t) = 1 + \sum_{n=1}^{\infty} [\mathrm{Hilb}^n X] t^n,$$

$$H^0_{\mathbb{A}^d}(t) = 1 + \sum_{n=1}^{\infty} [\mathrm{Hilb}^n_0 \mathbb{A}^d] t^n .$$

punctual Hilbert scheme

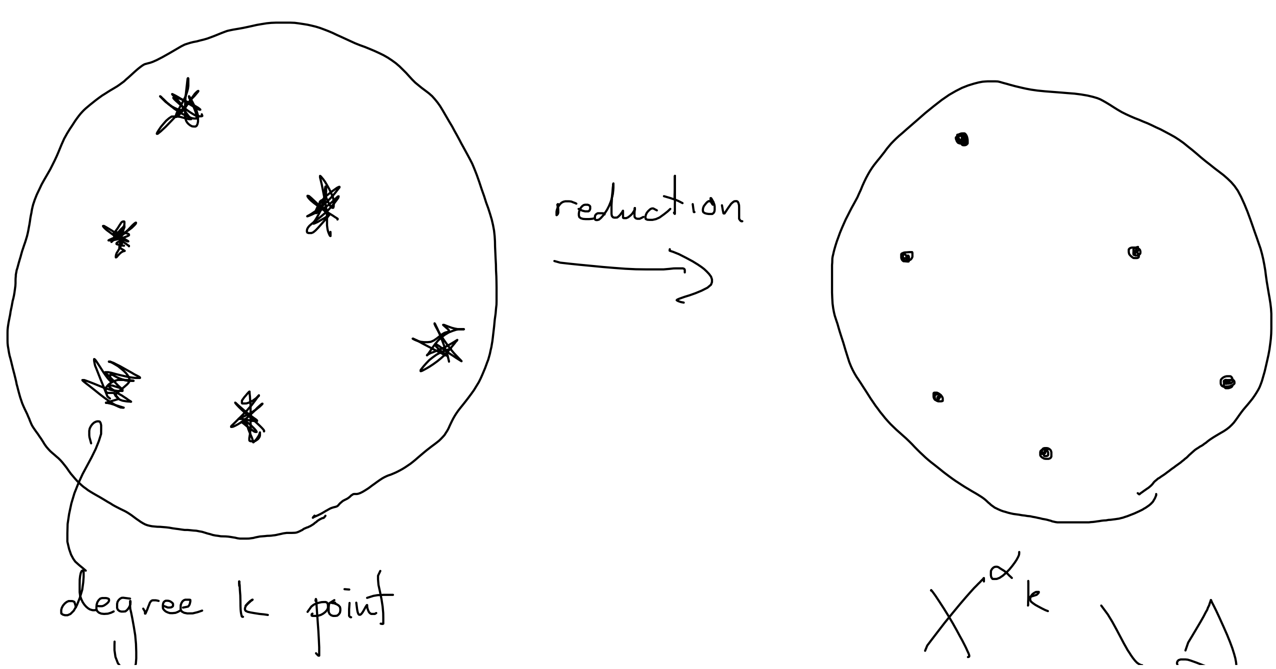
Then if $d = \dim X$, can show

$$H_X(t) = H^0_{\mathbb{A}^d}(t)^{[X]}$$

i.e. stratum $\mathrm{Hilb}^n_\alpha X$ of subschemes w/ α_i pts of multiplicity i is motivically what it should be:

The product over i of α_i degree i fuzzy points $([\mathrm{Hilb}^i_0 \mathbb{A}^d])$ supported at α_i distinct reduced points in X $(X^{\alpha_i} \setminus \Delta)$.

Picture: $\alpha = k^{\alpha_k}$, $\mathrm{Hilb}_\alpha^{k \cdot \alpha_k} X =$



degree k point

$$X^{\alpha_k} \setminus \Delta$$

$$(\text{mod } S_{\alpha_k})$$

$(\mathbb{C}^3)^{[n]}$ as a degeneracy locus

Theorem: $\mathrm{Hilb}^n\,\mathbb{C}^3$ can be described as the degeneracy locus $\{df_n = 0\}$ for some regular map $f_n : M_n \to \mathbb{C}$ where M_n is a smooth ambient variety.

Proof: We claim that $\mathrm{Hilb}^n\,\mathbb{C}^3$ has the following explicit description:

$$\left\{ (A,B,C,v) \;\middle|\; \begin{array}{l} A,B,C \in \mathrm{End}(\mathbb{C}^n) \\ v \in \mathbb{C}^n, \\ [A,B] = [A,C] = [B,C] = 0, \end{array} \right.$$

v should span $\mathbb{C}^n$ under polynomial action of A, B, C $\}$

$$/ GL_n(\mathbb{C})$$

$$g \cdot (A, B, C, v) = (gAg^{-1}, gBg^{-1}, gCg^{-1}, gv)$$

What's the identification?

Points of $\mathrm{Hilb}^n \mathbb{C}^3$ are just ideals $I \trianglelefteq \mathbb{C}[x, y, z]$ s.t.

$$\dim_{\mathbb{C}} (\mathbb{C}[x, y, z] / I) = n.$$

Given such an I, first identify $\mathbb{C}[x,y,z]/I \cong \mathbb{C}^n$ (then mod out by $GL_n(\mathbb{C})$ action to make canonical) and construct (A,B,C,v) as follows:

$$A, B, C := \text{respctively } x\cdot,\ y\cdot,\ z\cdot$$

$$v := 1 \in \mathbb{C}[x,y,z]/I$$

Conversely, given (A,B,C,v), recover $I \trianglelefteq \mathbb{C}[x,y,z]$ as:

$$\ker\left(\mathbb{C}[x,y,z] \ni f \mapsto f(A,B,C)v \right)$$

Can convince self that these two maps are well-defined and mutually inverse, so have set-theoretic equality (on $\mathbb{C}$-points). How to show scheme structure?

Just check functor of points explicitly (i.e functor $\mathcal{H}ilb^n \mathbb{C}^3$ is represented by the set, we will show later that set has scheme structure).

Call set H, then have flat family $\mathcal{H} \to H$ of 0-dimensional subschemes of $\mathbb{C}^3$ given by associating to $(A,B,C,v) \in H$

$$\ker\left(k[x,y,z] \ni f \mapsto f(A,B,C)v \right).$$

To check representability, suppose $\pi : Z \to U$ is flat family of 0-dimensional length n subschemes of $\mathbb{C}^3$. If $U = \bigcup_\lambda U_\lambda$ is open cover trivializing $\pi_* \mathcal{O}_Z$, can use multiplication by x, y, z to define

maps $\mathcal{U}_\chi \to \widetilde{H}$ (H w/o $/GL_n(\mathbb{C})$),

and these glue when modding out

by $GL_n(\mathbb{C})$ since the trivializations

of $\pi_* \mathcal{O}_Z$ on individual $\mathcal{U}_\chi$ are

related by $GL_n(\mathbb{C})$, so passes

to global unique (need to check)

map $\phi : \mathcal{U} \to H$ s.t.

$\phi^* \mathcal{H} = Z$, so H represents

the functor and hence is

$\mathrm{Hilb}^n \mathbb{C}^3$ if H is a scheme (which we will show). Nakajima has details of representability proof for $\mathbb{C}^2$.

Embedding of $\mathrm{Hilb}^n \mathbb{C}^3$ as critical locus:

Want to construct smooth variety M_n and regular map $f_n : M_n \to \mathbb{C}$ s.t. $H = \{df_n = 0\}$. This will give scheme structure of H and hence also prove that

$H = \mathrm{Hilb}^n \mathbb{C}^3$ as schemes.

Set $U_n \subseteq \mathrm{End}(\mathbb{C}^n)^{\oplus 3} \oplus \mathbb{C}^n$ to be the subset of (A, B, C, v) such that v spans $\mathbb{C}^n$ under iterated action by A, B, C.

This is an open stability condition: can be rewritten as "(A, B, C, v) s.t. there does not exist a proper subspace $S \subseteq \mathbb{C}^n$ s.t. $TS \subset S$ and

$v \in S$ for any of $T = A, B, C$".

Thus U_n is an open subset of an affine space. If have $GL_n(\mathbb{C}) \curvearrowright End(\mathbb{C}^n)^{\oplus 3} \oplus \mathbb{C}^n$ in usual way then U_n is set of GIT stable points for this action linearized by character

$$\chi : GL_n(\mathbb{C}) \longrightarrow \mathbb{C}^\times$$

$$g \longmapsto \det(g)$$

(can check explicitly via 1-PS).

Stability condition implies action of $GL_n(\mathbb{C})$ on U_n is free, so GIT quotient

$$M_n := U_n \,/\, GL_n(\mathbb{C})$$

is smooth quasiprojective variety.

H is cut out by the additional equations $[A,B] = [A,C] = [B,C] = 0$ (equations are conjugation-equivariant).

Define $\tilde{f}_n : U_n \longrightarrow \mathbb{C}$

$$(A,B,C,v) \mapsto \mathrm{Tr}([A,B]C)$$

This map is conjugation-invariant, so descends to regular map

$$f_n : M_n \to \mathbb{C}.$$

Working explicitly in coordinates shows that $df_n = 0$ if and only if

$$[A,B] = [A,C] = [B,C] = 0 :$$

Explicitly,

$$\mathrm{Tr}([A,B]C) = \sum_{i,k} \sum_j (A_{ij}B_{jk} - B_{ij}A_{jk})C_{ki}$$

so

$$\partial_{C_{ki}} \mathrm{Tr}([A,B]C) = \sum_j A_{ij}B_{jk} - B_{ij}A_{jk},$$

$$\partial_{C_{ki}} \operatorname{tr}(C[A,B]) = \sum_j (A_{ij}B_{jk} - B_{ij}A_{jk}),$$

the $(i,k)^{th}$ entry in $[A,B]$, so that $df_n = 0 \Longrightarrow [A,B] = 0$.

Index-juggling shows the same for $\partial_{A_{ij}}$, $\partial_{B_{ij}}$, so that $df_n = 0$ implies commutation and conversely.

This concludes the proof.

(What fails for $\mathbb{C}^4$ and higher?)

Virtual motive of $\mathrm{Hilb}^n \mathbb{C}^3$

Given a regular map $f: M \to \mathbb{C}$ with M quasiprojective satisfying certain torus-equivariance properties, can use motivic integration (whatever that means...) to define absolute motivic vanishing cycle of f and show it can be computed as

$$[\varphi_f] = \underbrace{[f^{-1}(1)]}_{\text{generic fiber}} - \underbrace{[f^{-1}(0)]}_{\text{central fiber}} \in \mathcal{M}_{\mathbb{C}}.$$

Suppose $Z = \{df = 0\} \subset X$, and set

$$[Z]_{vir} = \mathbb{L}^{-\frac{\dim X}{2}} [\varphi_f].$$

By

computing fibrewise Euler characteristics in terms of Milnor fibres, can show that

$$\chi([Z]_{vir}) = \chi_{vir}(Z).$$

(All of this is not easy and relies on lots of other work on motivic integration, arc spaces).

Upshot: For $Z = \{df = 0\}$ a moduli space of sheaves on a Calabi-Yau threefold (call f super-potential or global Chern-Simons functional), the associated DT invariant is $\chi_{vir}(Z)$, so $[Z]_{vir}$ is motivic DT invariant.

For $Z = \mathrm{Hilb}^n \mathbb{C}^3$ we've already explicitly computed super-potential, so use this to explicitly compute $[\mathrm{Hilb}^n \mathbb{C}^3]_{vir}$.

We already know $[\mathrm{Hilb}^n \mathbb{C}^3]_{vir}$ exists, and some arguments show that classes $[\mathrm{Hilb}^n_\alpha \mathbb{C}^3]_{vir}$ for every partition α of n such that

$$[\mathrm{Hilb}^n \mathbb{C}^3]_{vir} = \sum_{\alpha \vdash n} [\mathrm{Hilb}^n_\alpha \mathbb{C}^3]_{vir}.$$

Theorem: Let $Z_{\mathbb{C}^3}(t)$ be the (motivic) partition function $Z_{\mathbb{C}^3}(t) = \sum_{n \geq 0} [\mathrm{Hilb}^n \mathbb{C}^3]_{vir} t^n$.

Then $Z_{\mathbb{C}^3}(t) = \prod^{\infty} \prod^{m-1} (1 - \mathbb{L}^{2+k-m/2} t^m)^{-1}$.

$m=1 \quad k=0$

Proof sketch: Set

$$Y_n = \{(A,B,C,v) \mid \mathrm{Tr}([A,B]C)=0\}$$

$$Z_n = \{(A,B,C,v) \mid \mathrm{Tr}([A,B]C)=1\}$$

$$\subset \mathrm{End}(\mathbb{C}^n)^{\times 3} \times \mathbb{C}^n.$$

We have an isomorphism

$$\left(\mathrm{End}(\mathbb{C}^n)^{\times 3} \times \mathbb{C}^n\right) \setminus Y_n \cong \mathbb{C}^* \times Z_n$$

via $(A,B,C,v) \mapsto \left(\mathrm{Tr}([A,B]C), \dfrac{A}{\mathrm{Tr}([A,B]C)}, B, C\right).$

By scissor relations, this implies

$$(1-\mathbb{L})([Y_n]-[Z_n]) = \mathbb{L}^{3n^2+n} - \mathbb{L}[Y_n].$$

Set $Y_n = Y_n' \sqcup Y_n''$ where Y_n' is locus where $[A,B]=0$ and Y_n'' is complement. Set C_n to be variety of pairs of commuting matrices in $\mathrm{End}(\mathbb{C}^n)$.

Get projections $Y_n' \to C_n$

$$Y_n'' \to \{\mathbb{C}^{2n^2} \setminus C_n\}$$

with simple (affine) fibers, analyze to get relation $\omega_n = \mathbb{L}^{n(n+1)}[C_n]$, where

$$\omega_n = [Y_n] - [Z_n].$$

To add stability condition, define the (A,B,C)-span of $v \in \mathbb{C}^n$

for $A, B, C \in \mathrm{End}(\mathbb{C}^n)$ to be

orbit of v under $\mathbb{C}[x,y,z]$ acting

via A, B, C. This is a linear subspace,

so set

$X_n^k = \{(A, B, C, v)\} \mid \dim_{\mathbb{C}} (A,B,C)\text{-span of } v = k\}$,

$Y_n^k = Y_n \cap X_n^k$

$Z_n^k = Z_n \cap X_n^k$.

Ultimate goal is $[Y_n^n] - [Z_n^n]$.

Have fibration $Y_n^k \longrightarrow Gr(k,n)$

sending (A,B,C,v) to (A,B,C)-span of v.

Linear algebra and fibration / scissor

relations " give

$$\omega_n^k := [Y_n^k] - [Z_n^k]$$

$$= \mathbb{L}^{(n-k)(n+2k)} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{L}} [C_{n-k}]\, \omega_k^k .$$

Since $\omega_n^n = \omega_n - \sum_{k=0}^{n-1} \omega_n^k$ by stratification,

use formulas to get

$$\omega_n^n = \mathbb{L}^{n(n+1)} [C_n] - \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_{\mathbb{L}} \mathbb{L}^{(n-k)(n+2k)} [C_{n-k}]\omega_k^k .$$

Since $[\varphi_{f_n}] = [f_n^{-1}(1)] - [f_n^{-1}(0)]$

$$= \frac{[Z_n^n]}{[GL_n]} - \frac{[Y_n^n]}{[GL_n]}$$

$$= \frac{-\omega_n^n}{\mathbb{L}^{\binom{n}{2}}[n]_{\mathbb{L}}!} ,$$

We've basically computed $[\mathrm{Hilb}^n \mathbb{C}^3]_{vir}$ in terms of $[C_n]$, which we know by Feit-Fine. Some algebra of above relations and putting into generating series gives

$$C(t\mathbb{L}^{1/2}) = Z_{\mathbb{C}^3}(t)\, C(t\mathbb{L}^{-1/2})$$

where $C(t) = \sum_{n \geq 0} \frac{[C_n]}{[GL_n]} t^n$.

Solving for $Z_{\mathbb{C}^3}(t)$ and using Feit-Fine formula gives

$$Z_{\mathbb{C}^3}(t) = \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} \left(1 - \mathbb{L}^{2+k-m/2} t^m\right)^{-1},$$

as desired.

Fun fact: setting $\mathbb{L}^{\frac{1}{2}} = -1$ in $Z_{\mathbb{C}^3}(t)$ recovers the MacMahon function for 3d-partitions, which we should expect.

Recall that we defined power structure on $\mathcal{M}_{\mathbb{C}}$, defined in a way that

$$\left(\sum_{n \geq 0} [\mathrm{Hilb}_0^n \mathbb{A}^{\dim X}] t^n \right)^{[X]}$$

$$= \sum_{n \geq 0} [\mathrm{Hilb}^n X] t^n .$$

Since $\chi : \mathcal{M}_{\mathbb{C}} \to \mathbb{Z}$ is a homomorphism and the power structure is defined in terms of $+$ and $\cdot$ w/ $[\mathrm{Hilb}_0^n \mathbb{A}^d]$, can replace classes w/ motivic virtual classes, so for X a threefold,

$$Z_X(t) := \sum_{n \geq 0} [\mathrm{Hilb}^n X]_{vir}\, t^n$$

$$= Z_{\mathbb{C}^3,0}(t)^{[X]}$$

(motivic virtual partition function for punctual Hilbert scheme

Trick: know $Z_{\mathbb{C}^3}(t)$, set $X = \mathbb{C}^3$

in above to get

$$Z_{\mathbb{C}^3}(t) = Z_{\mathbb{C}^3,0}(t)^{[\mathbb{C}^3]}.$$

Solve for $Z_{\mathbb{C}^3,0}(t)$, and then compute formula for $Z_X(t) = Z_{\mathbb{C}^3,0}(t)^{[X]}$.

Theorem: If $\dim X = 3$,

$$Z_X(t) = \mathrm{Exp}\left(\frac{t[X]_{vir}}{(1-\mathbb{L}^{\frac{1}{2}}t)(1-\mathbb{L}^{-\frac{1}{2}}t)}\right).$$

This can be rewritten in a way that depends on $\dim X$ and gives the correct partition functions for $\dim X = 0, 1, 2$ (don't need virtual class since smooth).

Weird facts: This formula that depends on $\dim X$ computes correct virtual class of $[\mathrm{Hilb}^n X]$ for $n \leq 3$ in all dimensions.

Up to a sign, χ of the partition function for $\dim X = d$ gives MacMahon's conjecture for # of d-dimensional partitions (expected by localization), but this conjecture is known to be only asymptotically correct.

Göttsche-like formula:

$$\text{Let } M_\delta(t, q^{\frac{1}{2}}) = \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} \left(1 - q^{\delta + \frac{1}{2} + k - \frac{m}{2}} t^m\right)^{-1}$$

("refined MacMahon function"), then

applying W-polynomial homomorphism to partition function $Z_X(t)$ $(\dim X = 3)$

gives

$$WZ_X(t) = \prod_{d=0}^{6} M_{\frac{d-3}{2}}\left(-t, -q^{\frac{1}{2}}\right)^{(-1)^d b_d}$$

for X smooth projective threefold.